

# WHERE IS $f(z)/f'(z)$ UNIVALENT?

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**ABSTRACT.** Let  $\mathcal{S}$  denote the family of all univalent functions  $f$  in the unit disk  $\mathbb{D}$  with the normalization  $f(0) = 0 = f'(0) - 1$ . There is an intimate relationship between the operator  $P_f(z) = f(z)/f'(z)$  and the Danikas-Ruscheweyh operator  $T_f := \int_0^z (tf'(t)/f(t)) dt$ . In this paper we mainly consider the univalence problem of  $F = P_f$ , where  $f$  belongs to some subclasses of  $\mathcal{S}$ . Among several sharp results and non-sharp results, we also show that if  $f \in \mathcal{S}$ , then  $F \in \mathcal{U}$  in the disk  $|z| < r$  with  $r \leq r_6 \approx 0.360794$  and conjecture that the upper bound for such  $r$  is  $\sqrt{2} - 1$ .

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\mathcal{B}$  denote the class of analytic functions  $\omega(z)$  in the unit disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  such that  $\omega(0) = 0$  and  $|\omega(z)| < 1$  for  $z \in \mathbb{D}$ . If  $f, g$  are two analytic functions in  $\mathbb{D}$ , then we say that  $f$  is subordinate to  $g$ , written  $f \prec g$  or  $f(z) \prec g(z)$ , if there exists an  $\omega \in \mathcal{B}$  such that  $f(z) = g(\omega(z))$ . We also note that if  $g$  is univalent, then it is easy to show that  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(\mathbb{D}) \subset g(\mathbb{D})$ .

We consider the family  $\mathcal{A}$  of all functions  $f$  analytic in  $\mathbb{D}$  with the normalization  $f(0) = 0 = f'(0) - 1$ . By  $\mathcal{S}$ ,  $\mathcal{S} \subset \mathcal{A}$ , we denote the class of univalent functions in  $\mathbb{D}$ . Certain special subclasses of  $\mathcal{S}$  possess various remarkable features due to their geometrical properties. By  $\mathcal{C}$ ,  $\mathcal{K}$ , and  $\mathcal{S}^*$  we denote the subclasses of  $\mathcal{S}$  which consist of convex, close-to-convex, and starlike functions, respectively. For  $\beta \in [0, 1)$ , let  $\mathcal{S}^*(\beta)$  denote the usual normalized class of all (univalent) starlike functions of order  $\beta$ . Analytically,  $f \in \mathcal{S}^*(\beta)$  if  $f \in \mathcal{A}$  and satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\beta)z}{1 - z}, \quad z \in \mathbb{D}.$$

It is well-known that  $\mathcal{C} \subsetneq \mathcal{S}^*(1/2)$ , and  $\mathcal{S}^* := \mathcal{S}^*(0)$ . At this point it is interesting to note that a function belonging to  $\mathcal{S}^*(1/2)$  may not be convex in  $|z| < R$  for any  $R > \sqrt{2\sqrt{3} - 3} = 0.68\dots$ , see [8, Theorem 1]. We say that  $f \in \mathcal{A}$  is starlike in  $|z| < r$  (i.e. to say  $f \in \mathcal{S}^*$  in  $|z| < r$ ) for some  $0 < r \leq 1$ , if  $f(|z| < r)$  is starlike with respect to the origin. This means that the last subordination condition is satisfied for  $|z| < r$  instead of the full disk  $|z| < 1$ . Similar convention will be followed for other classes. We refer to [3, 4, 11] for a detailed discussion on these classes. Also

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let us introduce some notations and definitions as follows:

$$\begin{aligned}\mathcal{U} &= \{f \in \mathcal{A} : |U_f(z)| < 1 \text{ for } z \in \mathbb{D}\}, \quad U_f(z) = f'(z) \left( \frac{z}{f(z)} \right)^2 - 1, \\ \mathcal{C}(-1/2) &= \left\{ f \in \mathcal{A} : \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \text{ for } z \in \mathbb{D} \right\}, \text{ and} \\ \mathcal{G} &= \left\{ f \in \mathcal{A} : \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}, \text{ for } z \in \mathbb{D} \right\}.\end{aligned}$$

According to Aksestén's theorem [1] (see also [10]), the strict inclusion  $\mathcal{U} \subsetneq \mathcal{S}$  holds. Moreover,  $\mathcal{C}(-1/2) \subset \mathcal{K}$ , and functions in  $\mathcal{G}$  are proved to be starlike in  $\mathbb{D}$ , see for eg. [12, Example 1, Equation (16)]. See also [7] for further details and investigation on the class  $\mathcal{G}$ .

This article concerns with the operator

$$(1) \quad F(z) := P_f(z) = \frac{f(z)}{f'(z)}$$

for locally univalent functions  $f \in \mathcal{A}$ . The main problem is to consider the univalence and starlikeness of  $P_f$  when  $f$  belongs to some of the subclasses of  $\mathcal{S}$  defined above.

Among others our interest in the operator  $P_f$  arose from the fact that there exists an intimate relation between this one and the Danikas-Ruscheweyh ([2]) operator

$$(2) \quad T_f(z) := \int_0^z \frac{tf'(t)}{f(t)} dt = z + \sum_{n=1}^{\infty} \frac{n}{n+1} c_n(f) z^{n+1} \quad (f \in \mathcal{S}),$$

where  $c_n(f)$  ( $n \geq 1$ ) denote the logarithmic coefficients of  $f \in \mathcal{S}$  defined by

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f) z^n.$$

The conjecture that  $T_f \in \mathcal{S}$  for each  $f \in \mathcal{S}$  remains open.

The relation between (1) and (2) becomes obvious, when one considers the equivalent operators in the  $w$ -plane where  $w = f(z)$ . Let  $g(w) = f^{-1}(w)$  be the function inverse to  $f$ . If we transform the operator  $P_f$  to the  $w$ -plane, we get the operator

$$Q(g)(w) = wg'(w) = q(w).$$

A similar consideration concerning the Danikas-Ruscheweyh operator results in

$$S(g)(w) = \int_0^w \frac{g(u)}{u} du = s(w).$$

Now it is immediately seen that

$$Q^{-1}(q)(w) = \int_0^w \frac{q(u)}{u} du = S(q)(w) \quad \text{and} \quad S^{-1}(s)(w) = ws'(w) = Q(s)(w).$$

## 2. PRELIMINARIES AND TWO EXAMPLES

We remark that if  $f \in \mathcal{S}$  then  $(z/f(z)) \neq 0$  in  $\mathbb{D}$  and hence,  $f$  can be represented as Taylor's series of the form

$$(3) \quad f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}.$$

According to the well-known Area Theorem [4, Theorem 11 on p.193 of Vol. 2], for  $f \in \mathcal{S}$  of the form (3), one has

$$(4) \quad \sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1$$

but this condition is not sufficient for the univalence of  $f$ . On the other hand, if  $f \in \mathcal{A}$  of the form (3) satisfies the condition

$$(5) \quad \sum_{n=2}^{\infty} (n-1)|b_n| \leq 1,$$

then  $f \in \mathcal{U}$ . The condition (5) is also necessary if  $b_n \geq 0$  for  $n \geq 1$ . The constant 1 is the best possible in the sense that if

$$\sum_{n=2}^{\infty} (n-1)|b_n| = 1 + \varepsilon,$$

for some  $\varepsilon > 0$ , then there exists an  $f$  which is not univalent in  $\mathbb{D}$ .

Let us continue the discussion with two examples. Consider

$$f_1(z) = \frac{z(1 - \frac{z}{2})}{(1 - z)^2}, \quad \text{and} \quad f_2(z) = z - \frac{z^2}{2}.$$

Then  $f_1 \in \mathcal{C}(-1/2)$  and  $f_2 \in \mathcal{G}$ . Define

$$F_j(z) = P_{f_j}(z) = \frac{f_j(z)}{f'_j(z)}, \quad \text{for } j = 1, 2,$$

so that

$$F_1(z) = z - \frac{3}{2}z^2 + \frac{1}{2}z^3 \quad \text{and} \quad F_2(z) = \frac{z(1 - \frac{z}{2})}{1 - z}.$$

(1) We have that

$$F'_1(z) = \frac{3}{2}z^2 - 3z + 1 = \frac{3}{2}(z - r_+)(z - r_-), \quad r_{\pm} = 1 \pm \frac{\sqrt{3}}{3}$$

and therefore  $F'_1(r_-) = 0$ , where  $r_- = 1 - \frac{\sqrt{3}}{3} = 0.4226497 \dots$ . We claim that  $\operatorname{Re}(F'_1(z)) > 0$  for  $|z| < r_-$ . To do this, we observe that

$$\operatorname{Re}(F'_1(re^{i\theta})) = 3r^2 \cos^2 \theta - 3r \cos \theta + 1 - \frac{3}{2}r^2,$$

then it is easy to show that  $\operatorname{Re}(F'_1(re^{i\theta})) > 0$  for  $-1 \leq \cos \theta \leq 1$  and  $0 \leq r < r_-$ . It means that  $F_1$  is univalent in the disc  $|z| < r_-$ .

(2) It is a simple exercise to see that  $F_2 \in \mathcal{U}$ . In fact,

$$\frac{z}{F_2(z)} = \frac{1-z}{1-\frac{z}{2}} = 1 - \frac{\frac{z}{2}}{1-\frac{z}{2}} = 1 - \frac{z}{2} - \sum_{n=2}^{\infty} b_n z^n, \quad b_n = \frac{1}{2^n},$$

so that  $z/F_2(z)$  is non-vanishing in  $\mathbb{D}$  and thus,

$$-z \left( \frac{z}{F_2(z)} \right)' + \frac{z}{F_2(z)} - 1 = \left( \frac{z}{F_2(z)} \right)^2 F_2'(z) - 1 = \left( \frac{\frac{z}{2}}{1-\frac{z}{2}} \right)^2$$

from which we easily see that  $|U_{F_2}(z)| < 1$  for  $z \in \mathbb{D}$ . Indeed, by a direct computation, we see that the function  $w = (z/2)/(1 - (z/2))$  maps  $\mathbb{D}$  onto the disk  $|w - (1/3)| < 2/3$  so that  $w \in \mathbb{D}$  and thus,  $w^2 \in \mathbb{D}$ . This observation gives that  $|U_{F_2}(z)| < 1$  in  $\mathbb{D}$  and hence,  $F_2 \in \mathcal{U}$ . Alternately, using the series expansion for  $F_2$ , we find that

$$\sum_{n=2}^{\infty} (n-1)|b_n| = \sum_{n=2}^{\infty} (n-1)\frac{1}{2^n} = 1$$

and, by the sufficient condition (5), it follows that  $F_2 \in \mathcal{U}$ .

### 3. MAIN RESULTS

Let  $\omega \in \mathcal{B}$ . Then by the Schwarz lemma it follows that  $|\omega(z)| \leq |z|$  for  $z \in \mathbb{D}$  and by the Schwarz-Pick lemma we have

$$(6) \quad |\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2} \quad \text{for } z \in \mathbb{D}.$$

Clearly,  $\frac{\omega(z)}{z}$  is analytic in  $\mathbb{D}$  and  $|\omega(z)/z| \leq 1$  in  $\mathbb{D}$ . The Schwarz-Pick lemma, namely, (6), applied to  $\omega(z)/z$  shows that

$$(7) \quad |z\omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2}.$$

These three inequalities will be used frequently in the proof of our main results.

**Theorem 1.** *If  $f \in \mathcal{S}^*(\beta)$ , then  $P_f \in \mathcal{U}$  in the disk  $|z| < 1/(1 + \sqrt{2(1-\beta)})$ . The result is sharp (as for univalence) as the function  $z/(1-z)^{2(1-\beta)}$  shows.*

*Proof.* Each  $f \in \mathcal{S}^*(\beta)$  and  $F = P_f$  defined by (1) can be written as

$$\frac{zf'(z)}{f(z)} = \frac{1 + (1-2\beta)\omega(z)}{1 - \omega(z)} \quad \text{and} \quad F(z) = \frac{z(1 - \omega(z))}{1 + (1-2\beta)\omega(z)},$$

where  $\omega \in \mathcal{B}$ . Clearly,  $\frac{\omega(z)}{z}$  is analytic in  $\mathbb{D}$  and  $|\omega(z)/z| \leq 1$  in  $\mathbb{D}$ . Using the last two relations, we observe that

$$(8) \quad U_F(z) = -z \left( \frac{z}{F(z)} \right)' + \frac{z}{F(z)} - 1 = \frac{zf'(z)}{f(z)} - z \left( \frac{zf'(z)}{f(z)} \right)' - 1$$

and thus,

$$U_F(z) = 2(1-\beta) \left( \frac{\omega(z)}{1-\omega(z)} - \frac{z\omega'(z)}{(1-\omega(z))^2} \right) = 2(1-\beta) \left( \frac{(\omega(z) - z\omega'(z)) - \omega^2(z)}{(1-\omega(z))^2} \right)$$

from which and (7), we obtain that

$$\begin{aligned} |U_F(z)| &\leq 2(1-\beta) \left( \frac{|\omega(z) - z\omega'(z)|}{(1-|\omega(z)|)^2} + \frac{|\omega(z)|^2}{(1-|\omega(z)|)^2} \right) \\ &\leq 2(1-\beta) \left( \frac{\frac{|z|^2 - |\omega(z)|^2}{1-|z|^2}}{(1-|\omega(z)|)^2} + \frac{|\omega(z)|^2}{(1-|\omega(z)|)^2} \right) \\ &= \frac{2(1-\beta)|z|^2}{1-|z|^2} \left( \frac{1+|\omega(z)|}{1-|\omega(z)|} \right) \leq \frac{2(1-\beta)|z|^2}{1-|z|^2} \left( \frac{1+|z|}{1-|z|} \right) = \frac{2(1-\beta)|z|^2}{(1-|z|)^2} \end{aligned}$$

which can easily be seen to be less than 1 if  $|z| < 1/(1 + \sqrt{2(1-\beta)})$ . Thus,  $F$  belongs to  $\mathcal{U}$  in the disk  $|z| < 1/(1 + \sqrt{2(1-\beta)})$ .

To prove the sharpness part, we consider  $k_\beta(z) = z/(1-z)^{2(1-\beta)}$  and define

$$F_\beta(z) = P_{k_\beta}(z) = \frac{k_\beta(z)}{k'_\beta(z)}.$$

Then we see that  $k_\beta \in \mathcal{S}^*(\beta)$  and

$$F_\beta(z) = \frac{z(1-z)}{1+(1-2\beta)z} \quad \text{and} \quad \frac{z}{F_\beta(z)} = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta) \sum_{n=1}^{\infty} z^n.$$

Define  $G_\beta(z) = \frac{1}{r} F_\beta(rz)$  and observe that

$$\frac{z}{G_\beta(z)} = 1 + 2(1-\beta) \sum_{n=1}^{\infty} r^n z^n.$$

According to (5), the function  $G_\beta$  is in  $\mathcal{U}$  (and hence is univalent in  $\mathbb{D}$ ) if and only if

$$2(1-\beta) \sum_{n=2}^{\infty} (n-1)r^n \leq 1, \quad \text{i.e.} \quad \frac{2(1-\beta)r^2}{(1-r)^2} \leq 1.$$

This gives the condition  $0 < r \leq r_1 = 1/(1 + \sqrt{2(1-\beta)})$ . Thus, the function  $F_\beta$  is univalent in the disk  $|z| < r_1$  and not in any larger disk with center at the origin. Note also that

$$F'_\beta(z) = \frac{1-2z-(1-2\beta)z^2}{(1+(1-2\beta)z)^2}$$

and thus,  $F'_\beta(r_1) = 0$ . Moreover,

$$U_{F_\beta}(z) = \frac{1-2z-(1-2\beta)z^2}{(1-z)^2} - 1$$

showing that  $U_{F_\beta}(r_1) = -1$ . Thus, the number  $r_1$  is best both for univalence and also for  $\mathcal{U}$ . The proof is complete.  $\square$

**Corollary 1.** *If  $f \in \mathcal{S}^*$ , then  $P_f \in \mathcal{U} \cap \mathcal{S}^*$  in the disk  $|z| < \sqrt{2} - 1$ . The result is sharp (as for univalence) as the Koebe function  $z/(1-z)^2$  shows.*

*Proof.* It suffices to prove the starlikeness part since  $P_f \in \mathcal{U}$  follows from Theorem 1 by taking  $\beta = 0$ . Thus, for the proof of the second part, it suffices to observe by (6) that

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| -\frac{2z\omega'(z)}{1-\omega^2(z)} \right| \leq \frac{2|z||\omega'(z)|}{1-|\omega(z)|^2} \leq \frac{2|z|}{1-|z|^2}$$

which is again less than 1 provided  $|z| < \sqrt{2} - 1$ . In particular,  $F$  is starlike in the disk  $|z| < \sqrt{2} - 1$ . Sharpness part follows from the discussion in Theorem 1 with  $\beta = 0$ .  $\square$

**Corollary 2.** *If  $f \in \mathcal{S}^*(1/2)$ , then  $P_f \in \mathcal{U} \cap \mathcal{S}^*$  in the disk  $|z| < 1/2$ . The result is sharp as the function  $z/(1-z)$  shows.*

*Proof.* Choose  $\beta = 1/2$  in Theorem 1 and observe that it suffices to prove the starlikeness part. As in the proof of Theorem 1, for each  $f \in \mathcal{S}^*(1/2)$ , we have

$$\frac{zf'(z)}{f(z)} = \frac{1}{1-\omega(z)} \quad \text{and} \quad F(z) = z(1-\omega(z))$$

for some  $\omega \in \mathcal{B}$ . By (6) and the fact that  $|\omega(z)| \leq |z|$ , we obtain

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| \frac{-z\omega'(z)}{1-\omega(z)} \right| \leq \frac{|z||\omega'(z)|}{1-|\omega(z)|} \leq \frac{|z|(1+|\omega(z)|)}{1-|z|^2} \leq \frac{|z|(1+|z|)}{1-|z|^2} = \frac{|z|}{1-|z|}$$

which is less than 1 if  $|z| < 1/2$ . Note that for  $f(z) = z/(1-z)$ , one has  $F(z) = z - z^2$  and thus,  $|F'(z) - 1| = 2|z| < 1$  for  $|z| < 1/2$  and  $F'(1/2) = 0$ . Thus,  $F$  is univalent in the disk  $|z| < 1/2$  and not in any larger disk with center at the origin. Also, it is easy to see that  $F(z)$  is starlike for  $|z| < 1/2$ . The desired conclusion follows.  $\square$

**Corollary 3.** *If  $f \in \mathcal{S}^*(1/2)$  such that  $f''(0) = 0$ , then  $P_f$  is starlike in the disk  $|z| < r_2$ , where  $r_2 \approx 0.543689$  is the root of the equation  $\phi_2(r) = 0$ , where*

$$\phi_2(r) = r^3 + r^2 + r - 1.$$

*Proof.* Clearly, we just need to apply Corollary 2 with  $|\omega(z)| \leq |z|^2$ . This will lead to the inequality

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \leq \frac{|z|(1+|z|^2)}{1-|z|^2}$$

which is clearly less than 1 if  $|z|^3 + |z|^2 + |z| - 1 < 0$ . The result follows.  $\square$

**Corollary 4.** *Let  $f$  belong to either  $\mathcal{S}^*(1/2)$  or  $\mathcal{C}(-1/2)$ , such that  $f''(0) = 0$ . Then  $F \in \mathcal{U}$  in the disk  $|z| < 1/\sqrt{3}$ .*

*Proof.* It known that [9, p. 68] if  $\mathcal{C}(-1/2)$  with  $f''(0) = 0$ , then  $f \in \mathcal{S}^*(1/2)$ . In view of this result, it suffices to prove the corollary when  $f$  belongs to  $\mathcal{S}^*(1/2)$  with  $f''(0) = 0$ . However, using the proof of Theorem 1 with  $\beta = 1/2$  and  $|\omega(z)| \leq |z|^2$ , we easily obtain that

$$|U_F(z)| \leq \frac{|z|^2}{1 - |z|^2} \left( \frac{1 + |\omega(z)|}{1 - |\omega(z)|} \right) \leq \frac{|z|^2}{(1 - |z|^2)} \left( \frac{1 + |z|^2}{1 - |z|^2} \right)$$

which is less than 1 provided  $1 - 3|z|^2 > 0$  and this gives the disk  $|z| < 1/\sqrt{3}$ . The proof is complete.  $\square$

A locally univalent function  $f \in \mathcal{A}$  is said to belong to  $\mathcal{G}(\alpha)$ , for some  $\alpha \in (0, 1]$ , if it satisfies the condition

$$(9) \quad \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{\alpha}{2}, \quad z \in \mathbb{D}.$$

Thus, we have  $\mathcal{G} := \mathcal{G}(1)$ .

**Theorem 2.** *If  $f \in \mathcal{G}(\alpha)$  for some  $\alpha \in (0, 1]$ , then  $P_f$  is starlike in the disk  $|z| < 1 + \alpha - \sqrt{\alpha(1 + \alpha)}$ .*

*Proof.* Let  $f \in \mathcal{G}(\alpha)$  and  $F$  be given by (1). Then we have (see eg. [5, Theorem 1])

$$\frac{zf'(z)}{f(z)} \prec \frac{(1 + \alpha)(1 - z)}{1 + \alpha - z}, \quad z \in \mathbb{D},$$

and thus, we may write

$$\frac{zf'(z)}{f(z)} = \frac{(1 + \alpha)(1 - \omega(z))}{1 + \alpha - \omega(z)} \quad \text{and} \quad F(z) = P_f = \frac{z(1 + \alpha - \omega(z))}{(1 + \alpha)(1 - \omega(z))}$$

for some  $\omega \in \mathcal{B}$ . By a computation, we obtain that

$$\frac{zF'(z)}{F(z)} - 1 = \frac{\alpha z \omega'(z)}{(1 - \omega(z))(1 + \alpha - \omega(z))}$$

and, as before, it follows from the Schwarz-Pick lemma that

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \leq \frac{\alpha |z| |\omega'(z)|}{(1 + \alpha - |\omega(z)|)(1 - |\omega(z)|)} \leq \frac{\alpha |z|}{(1 + \alpha - |z|)(1 - |z|)}$$

which is less than 1 provided  $\phi_3(|z|) > 0$ , where  $\phi_3(r) = r^2 - 2(1 + \alpha)r + 1 + \alpha$ . Thus, we conclude that  $P_f$  is starlike in the disk  $|z| < r_3(\alpha) = 1 + \alpha - \sqrt{\alpha(1 + \alpha)}$ , where  $r_3(\alpha)$  is the root of the equation  $\phi_3(r) = 0$  in the interval  $(0, 1]$ . The theorem follows.  $\square$

Taking  $\alpha = 1$  gives

**Corollary 5.** *If  $f \in \mathcal{G}$ , then  $P_f$  is starlike in the disk  $|z| < 2 - \sqrt{2} \approx 0.585786$ .*

The same reasoning gives as in Corollary 3 the following.

**Corollary 6.** *If  $f \in \mathcal{G}(\alpha)$  such that  $f''(0) = 0$  and for some  $\alpha \in (0, 1]$ , then  $P_f$  is starlike in  $|z| < r_4(\alpha)$ , where  $r_4(\alpha)$  is the root in the interval  $(0, 1]$  of the equation  $\phi_4(r) = 0$ ,*

$$\phi_4(r) = r^4 - \alpha r^3 - (2 + \alpha)r^2 - \alpha r + 1 + \alpha.$$

*Proof.* In this case, the corresponding inequality for  $f \in \mathcal{G}(\alpha)$  in Theorem 2 becomes

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \leq \frac{\alpha|z|}{1 - |z|^2} \left( \frac{1 + |\omega(z)|}{1 + \alpha - |\omega(z)|} \right) \leq \frac{\alpha|z|}{1 - |z|^2} \left( \frac{1 + |z|^2}{1 + \alpha - |z|^2} \right)$$

which is less than 1 if  $\phi_4(|z|) > 0$ . The result follows.  $\square$

Setting  $\alpha = 1$  gives

**Corollary 7.** *If  $f \in \mathcal{G}$  such that  $f''(0) = 0$ , then  $P_f$  is starlike in  $|z| < r_4$ , where  $r_4 \approx 0.64731$  is the root in the interval  $(0, 1]$  of the equation  $r^4 - r^3 - 3r^2 - r + 2 = 0$ .*

**Theorem 3.** *If  $f \in \mathcal{G}(\alpha)$  for some  $\alpha \in (0, 1]$ , then  $F \in \mathcal{U}$  in the disk  $|z| < r_5(\alpha)$ , where  $r_5(\alpha) = \sqrt{\frac{-\alpha + \sqrt{(1+\alpha)^2 + 1}}{2}}$ .*

*Proof.* Let  $f \in \mathcal{G}(\alpha)$  and  $F = P_f$  be given by (1). Then, following the proof of Theorem 2, one has

$$\frac{z}{F(z)} - 1 = -\frac{\alpha\omega(z)}{1 + \alpha - \omega(z)}$$

and, using this relation, we find that

$$\begin{aligned} U_F(z) &= -\frac{\alpha\omega(z)}{1 + \alpha - \omega(z)} + \frac{\alpha(1 + \alpha)z\omega'(z)}{(1 + \alpha - \omega(z))^2} \\ &= \frac{\alpha[(1 + \alpha)(z\omega'(z) - \omega(z)) + \omega^2(z)]}{(1 + \alpha - \omega(z))^2} \end{aligned}$$

so that, by (7), we easily have as before that

$$\begin{aligned} |U_F(z)| &\leq \frac{\alpha}{(1 + \alpha - |\omega(z)|)^2} \left( (1 + \alpha) \left( \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2} \right) + |\omega(z)|^2 \right) \\ &= \frac{\alpha}{1 - |z|^2} \left( \frac{-(\alpha + |z|^2)|\omega(z)|^2 + (1 + \alpha)|z|^2}{(1 + \alpha - |\omega(z)|)^2} \right) = \frac{\alpha\phi(t)}{1 - r^2}, \end{aligned}$$

where we put  $|z| = r$ ,  $|\omega(z)| = t$  and

$$\phi(t) = \frac{-(\alpha + r^2)t^2 + (1 + \alpha)r^2}{(1 + \alpha - t)^2}, \quad 0 \leq t \leq r.$$

We compute that

$$\phi'(t) = \frac{2(1 + \alpha)}{(1 + \alpha - t)^3} [-(\alpha + r^2)t + r^2],$$



and it is easy to see that  $\phi$  attains its maximum value  $\phi(t_0)$ , where  $t_0 = \frac{r^2}{\alpha+r^2}$  and  $\phi''(t_0) < 0$ . A calculation gives

$$\phi(t_0) = \frac{r^2(\alpha + r^2)}{\alpha(1 + \alpha + r^2)}$$

and thus, we have

$$|U_F(z)| \leq \frac{\alpha\phi(t_0)}{1-r^2} = \frac{r^2(\alpha + r^2)}{(1-r^2)(1 + \alpha + r^2)}$$

which is less than 1 if  $2r^4 + 2\alpha r^2 - (1 + \alpha) < 0$ . This gives that  $|U_F(z)| < 1$  for  $0 < r \leq r_5(\alpha)$ , where  $r_5(\alpha)$  is the root of the equation  $2r^4 + 2\alpha r^2 - (1 + \alpha) = 0$ , that lies in the interval  $(0, 1)$ . The conclusion follows.  $\square$

The choice  $\alpha = 1$  yields the following.

**Corollary 8.** *If  $f \in \mathcal{G}$ , then  $F$  belongs to the class  $\mathcal{U}$  in the disk  $|z| < \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$ .*

**Theorem 4.** *Let  $f \in \mathcal{S}$  with  $a_2 = f''(0)/2!$ . Then  $F$  belongs to  $\mathcal{U}$  in the disk  $|z| < r_6(|a_2|)$ , where  $r_6(|a_2|)$  is the root of the equation  $\phi_5(r) = 0$  that lies in the interval  $(0, 1)$ , where*

$$\phi_5(r) = (a+1-\frac{1}{4}b^2)r^{10} - (5a+5-\frac{5}{4}b^2)r^8 + (19a+10-\frac{19}{4}b^2)r^6 + (9a-10-\frac{9}{4}b^2)r^4 + 5r^2 - 1$$

with  $b = |a_2|$  and  $a = \frac{2\pi^2-12}{3} \approx 2.57974$ .

*Proof.* Let  $f \in \mathcal{S}$  and following the idea of [6, Theorem 4], we consider

$$(10) \quad \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f) z^n,$$

where  $c_n(f)$  ( $n \geq 1$ ) denote the logarithmic coefficients of  $f$  with  $c_1(f) = a_2$ . Further, for  $f \in \mathcal{S}$  the following sharp inequality is known from the work of Roth [13, Theorem 1.1]

$$\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^2 |c_n(f)|^2 \leq \frac{2\pi^2 - 12}{3} = a.$$

By (10), we obtain

$$\frac{zf'(z)}{f(z)} - 1 = \sum_{n=1}^{\infty} n c_n(f) z^n$$

which by the relation (8) gives that

$$U_F(z) = - \sum_{n=1}^{\infty} n(n-1) c_n(f) z^n$$

Values of $ a_2 $	values of $r_6( a_2 )$	Values of $ a_2 $	values of $r_6( a_2 )$
0.25	0.361166	1.25	0.370874
0.5	0.362294	1.5	0.375923
0.75	0.364226	1.75	0.382504
1	0.367042	2	0.391124

TABLE 1. Values of  $r_6(|a_2|)$  for different values of  $|a_2|$ 

and thus, by the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
|U_F(z)| &= \left| \sum_{n=2}^{\infty} n(n-1)c_n(f)z^n \right| \\
&\leq \left( \sum_{n=2}^{\infty} \left( \frac{n}{n+1} \right)^2 |c_n(f)|^2 \right)^{\frac{1}{2}} \left( \sum_{n=2}^{\infty} (n^2-1)^2 |z|^{2n} \right)^{\frac{1}{2}} \\
&\leq \left( a - \frac{1}{4}|c_1(f)|^2 \right)^{\frac{1}{2}} \left( \frac{|z|^4(|z|^6 - 5|z|^4 + 19|z|^2 + 9)}{(1-|z|^2)^5} \right)^{\frac{1}{2}}
\end{aligned}$$

which is less than 1 whenever,

$$\left( a - \frac{1}{4}|c_1(f)|^2 \right) |z|^4(|z|^6 - 5|z|^4 + 19|z|^2 + 9) < (1-|z|^2)^5.$$

If we put  $r = |z|$ , then the last inequality is equivalent to  $\phi_5(r) := \phi_5(r, |a_2|) < 0$ , where  $\phi_5(r)$  is as in the statement. The desired result follows.  $\square$

**Corollary 9.** *Let  $f \in \mathcal{S}$  with  $f''(0) = 0$ , and  $a = \frac{2\pi^2-12}{3}$ . Then  $F$  belongs to  $\mathcal{U}$  in the disk  $|z| < r_6$ , where  $r_6 \approx 0.360794$  is the root of the equation*

$$(a+1)r^{10} - 5(a+1)r^8 + (19a+10)r^6 + (9a-10)r^4 + 5r^2 - 1 = 0,$$

*that lies in the interval  $(0, 1)$ .*

*Proof.* Set  $a_2 = 0$  in Theorem 4.  $\square$

It is a simple exercise to see that the values  $r_6(|a_2|)$ , as the roots of the equation  $\phi_5(r) = 0$ , increase with increasing values of  $|a_2| \in [0, 2]$ . For a ready reference, we included in Table 1 a list of values of  $r_6(|a_2|)$  for certain choices of  $|a_2|$ . This observation shows that if  $f \in \mathcal{S}$ , then  $F \in \mathcal{U}$  in the disk  $|z| < r$  and the lower bound for  $r$  by Corollary 9 is  $r_6 \approx 0.360794$ . We end the discussion with a conjecture that the upper bound for the value of  $r$  is  $\sqrt{2}-1$  which is attained by the Koebe function.

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